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A Q-ANALOGUE OF HYPERGEOMETRIC SERIES BY Q-EXPONENTIAL FUNCTIONS

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ABSTRACT. We establish some new transformation involving bilateral hypergeometric series and summation formulae by q-exponential functions and mentioned two special cases of it.

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KEYWORDS AND PHRASES. Summation formula, Basic hypergeometric series, Bilateral hypergeometric series, q-exponential function.

1. Introduction

The study of basic hypergeometric series started in 1748 by Euler. The basic hypergeometric series was first studied systematically by Heine [8], but many early results are attributed to Euler, Gauss and Jacobi. Hypergeometric series find applications in various field of mathematics and physics, including probability theory, combinatorial identities, fluid dynamics and quantum mechanics and also basic hypergeometric series are now major interactions with Lie algebras, combinatorics, special functions and number theory, primarily due to their ability to represent complex functions and solve differential equations that arises in these domains. Using different parameters in bilateral hypergeometric series, one can derive various summation identities involving sums over integers, including identities related to binomial coefficients, Fibonacci number, and other combinatorial sequences. Many mathematicians Andrew[1], Bailey[3, 2], Dogoull[6], Gasper[7], Ismail[9], Jackson[11, 10], Somashekara[15] and Slater[14] contributed to the summation and transformation of hypergeometric series identities. For more details about hypergeometric series and its applications see[15], [13], [10], [12], [4], [1] and [2].

Before, we going to prove main results, we introduce by presenting some standard definition and identities, for complex numbers a and q and $|q| < 1$, Pochhammar Symbol[7] is defined as

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),$$

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

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and

$$(a)_n := (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1}$$

It is easy to deduce the following identity;

$$(a)_{s+t} = (a)_s(aq^s)_t, \tag{2}$$

where s and t are non negative integers.
The basic hypergeometric series [7] is defined as

$$_j\phi_k(a_1,a_2,\cdots,a_j;b_1,b_2,\cdots,b_k;q,z) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n(a_3;q)_n\cdots(a_j;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_k;q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+k-j}z^n,$$

where $q \neq 0$, which converges for $|z| < \infty$ if $j \leq k$ and for $|z| < 1$ if $j = k+1$.
The bilateral hypergeometric series[7] is defined as

$$_j\psi_k(a_1,a_2,\cdots,a_j;b_1,b_2,\cdots,b_k;q,z) = \sum_{n=-\infty}^{\infty}\frac{(a_1;q)_n(a_2;q)_n(a_3;q)_n\cdots(a_j;q)_n}{(b_1;q)_n(b_2;q)_n(b_3;q)_n\cdots(b_k;q)_n}[(-1)^n q^{\frac{n(n-1)}{2}}]^{k-j}z^n,$$

where which converges for $|\frac{b_1b_2b_3\ldots b_k}{a_1a_2a_3\ldots a_j}| < |z| < 1$ if $j = k$ and for $k > j$ it converges in the whole complex plane i.e for $|z| < \infty$.

Ramanujan's general theta function $f(a,b)$ [5, 12]is given by

$$\begin{aligned} f(a,b) &= 1 + \sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}(a^n+b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}, \ |ab| < 1. \end{aligned}$$

The q-binomial theorem[7]is given by

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{3}$$

and an interesting special case of q-exponential functions [7], which are due to Euler are given by

$$\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}z^n = \frac{1}{(z;q)_\infty}. \tag{4}$$

Jacobi's well-known triple product identity[7]

$$(zq^{\frac{1}{2}}, q^{\frac{1}{2}}/z, q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}z^n \tag{5}$$

can be easily derived by using Hein's summation formula[7].
The following lemma was proved in 1970[1] which is used to prove our main results;

Lemma 1.1. [15]*: Under the suitable conditions of convergence, if* $c_n = \sum_{m=0}^{\infty} a_{m+n}b_m$*, then,*

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} c_n. \tag{6}$$

2. New transformation of bilateral hypergeometric series.

We prove some new transformation for bilateral basic hypergeometric series in this section.

Theorem 2.1. *We have*

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(-zq^{n+1};q^2)_n} = \frac{(q^2,-zq,-q/z,-zq^{2n+1};q^2)_\infty}{(-1;q^2)_\infty}, \quad |z|<1,$$

where q is a complex number with $|q|<1$.

Proof. Let

$$a_n = q^{2n^2} z^n, \quad b_n = \frac{(-1)^n}{(q^4;q^4)_n}.$$

Then,

$$\begin{aligned} c_n &= \sum_{m=0}^{\infty} a_{m+n} b_m \\ &= \sum_{m=0}^{\infty} z^{m+n} q^{2(m+n)^2} \frac{(-1)^m}{(q^4;q^4)_m} \\ &= \sum_{m=0}^{\infty} z^n z^m q^{2m^2+4mn+2n^2} \frac{(-1)^m}{(q^4;q^4)_m} \\ &= z^n q^{2n^2} (-zq^{4n+2};q^4)_\infty. \end{aligned}$$

Using (6), we get

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(q^4;q^4)_m} \sum_{n=-\infty}^{\infty} (q^2)^{n^2} z^n = \sum_{n=-\infty}^{\infty} z^n q^{2n^2} (-zq^{4n+2};q^4)_\infty,$$

now using (4), Jacobi Product[7]and by replacing q by q^2, we get

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(-zq^{n+1};q^2)_n} = \frac{(q^2,-zq,-q/z,-zq^{2n+1};q^2)_\infty}{(-1;q^2)_\infty}, \quad |z|<1. \tag{7}$$

□

Theorem 2.2. *We have*

$$\sum_{n=-\infty}^{\infty} q^{2n} (q^{3(n+1)};q^3)_\infty = \frac{(q;q^3)_\infty}{(q^2;q^3)_\infty}, \quad |q|<1,$$

where q is a complex number.

Proof. Let

$$a_n = q^n z^n, \quad b_n = \frac{z^n}{(q^3;q^3)_n}.$$

Then,

$$\begin{aligned}
c_n &= \sum_{m=0}^{\infty} a_{m+n} b_m \\
&= \sum_{m=0}^{\infty} q^{m+n} z^{m+n} \frac{z^m}{(q^3;q^3)_m} \\
&= \sum_{m=0}^{\infty} q^m q^n z^m z^n \frac{z^m}{(q^3;q^3)_m} \\
&= q^n z^n \sum_{m=0}^{\infty} \frac{(qz^2)^m}{(q^3;q^3)_m},
\end{aligned}$$

using (4), we get

$$c_n = q^n z^n \frac{1}{(qz^2;q^3)_\infty},$$

replacing a by qz^2 in equation (1.1), we get

$$c_n = q^n z^n \frac{1}{(qz^2;q^3)_n (z^2 q^{3n+1};q^3)_\infty}.$$

Using equation (6), we get

$$\sum_{m=0}^{\infty} \frac{z^m}{(q^3;q^3)_m} \sum_{n=-\infty}^{\infty} q^n z^n = \sum_{n=-\infty}^{\infty} q^n z^n \frac{1}{(qz^2;q^3)_n (z^2 q^{3n+1};q^3)_\infty},$$

now using (4), we get

$$\frac{1}{(z;q^3)_\infty} \sum_{n=-\infty}^{\infty} q^n z^n = \sum_{n=-\infty}^{\infty} \frac{q^n z^n}{(qz^2;q^3)_n (z^2 q^{3n+1};q^3)_\infty},$$

then

$$\sum_{n=-\infty}^{\infty} q^n z^n (z^2 q^{3n+1};q^3)_\infty = (z;q^3)_\infty \sum_{n=-\infty}^{\infty} \frac{q^n z^n}{(qz^2;q^3)_n},$$

now replacing z by q, we get

$$\sum_{n=-\infty}^{\infty} q^{2n} (q^{3n+3};q^3)_\infty = (q;q^3)_\infty \sum_{n=-\infty}^{\infty} \frac{(q^2)^n}{(q^3;q^3)_n}.$$

Now, again using equation (4), we get

$$\sum_{n=-\infty}^{\infty} q^{2n} (q^{3n+3};q^3)_\infty = \frac{(q;q^3)_\infty}{(q^2;q^3)_\infty}.$$

□

Theorem 2.3. *We have*

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(zq;q^4)_n} = \frac{(q^2,-zq,-q/z;q^4)_\infty}{(-1,zq;q^4)_\infty}, \quad |z|<1, \tag{8}$$

where q is complex number with $|q|<1$.

Proof. Let

$$a_n = z^n q^{n^2}, \quad b_n = \frac{(-1)^n}{(q^4;q^4)_n}.$$

Then,

$$\begin{aligned} c_n &= \sum_{m=0}^{\infty} a_{m+n} b_m \\ &= \sum_{m=0}^{\infty} z^{m+n} q^{(m+n)^2} \frac{(-1)^m}{(q^4;q^4)_m} \\ &= z^n q^{n^2} \sum_{m=0}^{\infty} \frac{(q^2)^{\frac{m(m-1)}{2}} (-zq^{2n+1})^m}{(q^4;q^4)_m}, \\ &= z^n q^{n^2} (-zq^{2n+1};q^4)_\infty. \end{aligned}$$

Using (6), we get

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(q^4;q^4)_m} \sum_{n=-\infty}^{\infty} q^{n^2} z^n = \sum_{n=-\infty}^{\infty} z^n q^{n^2} (zq^{2n+1};q^4)_\infty.$$

Using (4) and Jacobi triple product[15], we get

$$\frac{1}{(-1;q^4)_\infty} (q^2, -zq, -q/z; q^4)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{n^2} (zq^{2n+1};q^4)_\infty. \tag{9}$$

Now replacing a by zq in (1), we get

$$(zq^{2n+1};q^4)_\infty = \frac{(zq;q^4)_\infty}{(zq;q^4)_n}, \tag{10}$$

using (10) in (9), we get

$$\frac{1}{(-1;q^4)_\infty} (q^2, -zq, -q/z; q^4)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{n^2} \frac{(zq;q^4)_\infty}{(zq;q^4)_n},$$

which implies

$$\sum_{n=-\infty}^{\infty} \frac{z^n q^{n^2}}{(zq;q^4)_n} = \frac{(q^2, -zq, -q/z; q^4)_\infty}{(-1, zq; q^4)_\infty}.$$

□

3. Some specific cases

In this part, we derive some specific cases of our Theorem 2.1 and Theorem 2.3.

(1) Replacing z by q in (7), we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q^{n+2};q^2)_n} = (q^4;q^4)_\infty (-q^{2n+2};q^2)_\infty.$$

(2) Replacing z by q in (8), we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}}{(q^2;q^4)_n} = (-q^2;q^4)_\infty.$$

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